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STABILITY THEOREMS FOR PROJECTIONS OF CONVEX SETS

BY

H. GROEMER†

Department of Mathematics, The University of Arizona, Tucson, AZ 85721, USA

ABSTRACT

Let $\mathscr{H}^{n-1}$ denote the set of all $(n-1)$-dimensional linear subspaces of euclidean n-dimensional space E^n ($n \geqq 3$), and let J and K be two compact convex subsets of E^n. It is well-known that J and K are translation equivalent (or homothetic) if for all $H \in \mathscr{H}^{n-1}$ the respective orthogonal projections, say J_H, K_H, are translation equivalent (or homothetic). This fact gives rise to the following stability problem: If $\varepsilon \geqq 0$ is given, and if for every $H \in \mathscr{H}^{n-1}$ a translate (or homothetic copy) of K_H is within Hausdorff distance ε of J_H, how close is J to a nearest translate (or homothetic copy) of K? In the case of translations it is shown that under the above assumptions there is always a translate of K within Hausdorff distance $(1+2\sqrt{2})\varepsilon$ of J. Similar results for homothetic transformations are proved and applications regarding the stability of characterizations of centrally symmetric sets and spheres are given. Furthermore, it is shown that these results hold even if $\mathscr{H}^{n-1}$ is replaced by a rather small (but explicitly specified) subset of $\mathscr{H}^{n-1}$.

1. Introduction

Let $\mathscr{C}^n$ denote the class of all nonempty compact convex subsets of n-dimensional euclidean space E^n, and $\mathscr{H}^m$ the class of all m-dimensional linear subspaces of E^n. Furthermore, if $K \in \mathscr{C}^n$, $H \in \mathscr{H}^{n-1}$ let K_H denote the orthogonal projection of K onto H. If $n \geqq 3$ and $J, K \in \mathscr{C}^n$ it is well-known and easily proved (cf. Leichtweiss [6, p. 241]) that the translation equivalence of J_H and K_H (for all $H \in \mathscr{H}^{n-1}$) implies the translation equivalence of J and K. If $d(J, K)$ denotes the Hausdorff distance and if we define the *translative distance* between J and K by

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(1) $$d_t(J,K) = \inf_{p \in E^n} d(J, K+p),$$

then this fact can also be expressed by saying that $d_t(J,K) = 0$ if $d_t(J_H, K_H) = 0$ (for all $H \in \mathscr{H}^{n-1}$). In this article we are concerned with the corresponding stability problem: If for some $\varepsilon \geqq 0$ and every $H \in \mathscr{H}^{n-1}$ it is known that $d_t(J_H, K_H) \leqq \varepsilon$, what can be said about the size of $d_t(J,K)$? Golubyatnikov [2] has announced that under the above assumption $d_t(J,K) \leqq \varepsilon$. However, we give an example (at the end of the proof of Theorem 1) which shows that this inequality cannot hold for all $J, K \in \mathscr{C}^n$. We are able to prove that $d_t(J,K) \leqq (1 + 2\sqrt{2})\varepsilon$ and that Golubyatnikov's claim is justified if J and K are centrally symmetric. These results are formulated in the following section as Theorem 1, and are used to establish several consequences regarding the stability of certain characterizations of centrally symmetric sets.

We also investigate analogous problems for (positively) homothetic transformations. It is known that J and K must be homothetic if for every $H \in \mathscr{H}^{n-1}$ ($n \geqq 3$) the projections J_H and K_H are homothetic (see [3], [5], [7] and the references cited in these articles). To obtain corresponding stability theorems one has to introduce suitable deviation measures. There are several possibilities but none of these is as natural as d_t in the case of translations. Definitions of the measure to be used here are given in Section 3. Theorems 2 and 3 contain our stability results based on these definitions. Several consequences regarding the case when one of the sets J, K is a ball are formulated as a corollary.

There is another aspect of our stability results that deserves mentioning. Already Hadwiger [5] has noted that J and K are homothetic if J_H and K_H are homothetic for all hyperplanes H belonging to some suitable subset of $\mathscr{H}^{n-1}$. The dimension of these subsets, as determined by the points on the unit sphere in E^n corresponding to the orthogonal unit vectors of the hyperplanes, is $n-2$. Our results show that actually certain one-dimensional sets (consisting of a great circle and two additional points) can be used for that purpose.

2. Stability results concerning translations

The m-dimensional linear subspaces of E^n (i.e. the members of $\mathscr{H}^m$) will simply be called *m-spaces*, and whenever the term "projection" is used it means the orthogonal projection onto some (implicitly or explicitly specified) m-space. If $H \in \mathscr{H}^m$ then $H^\perp$ denotes the orthogonal complement of H. A set $\mathscr{G} \subset \mathscr{H}^{n-1}$ will be said to be *full* if $\bigcup_{H \in \mathscr{G}} H^\perp$ contains at least a 2-space, say F, and a line orthogonal to F (i.e. a line contained in $F^{\perp}$). In other words, $\mathscr{G}$ is full exactly if the set of all orthogonal unit vectors of the planes $H \in \mathscr{G}$ determines on the unit sphere in E^n a set that contains a great circle and two corresponding antipodal "poles". Because of its repeated appearance it is convenient to introduce a special symbol for the constant $1 + 2\sqrt{2}$. We set

$$\beta = 1 + 2\sqrt{2}. \tag{2}$$

Using these definitions we can now formulate our stability results concerning the translative distance d_t.

THEOREM 1. *Let $\mathscr{G}$ be a full subset of $\mathscr{H}^{n-1}$ $(n \geqq 3)$ and $J, K \in \mathscr{C}^n$. If $\varepsilon \geqq 0$ and if for all $H \in \mathscr{G}$*

$$d_t(J_H, K_H) \leqq \varepsilon, \tag{3}$$

then

$$d_t(J, K) \leqq \beta\varepsilon. \tag{4}$$

The constant β in (4) *cannot be replaced by* 1, *even if $\mathscr{G} = \mathscr{H}^{n-1}$; but if both J and K are centrally symmetric, then* (3) *implies*

$$d_t(J, K) \leqq \varepsilon. \tag{5}$$

Before we give the proof of this theorem we add several remarks and two corollaries.

Although β in (4) cannot be replaced by 1 it is unlikely that $\beta = 1 + 2\sqrt{2}$ is best possible. To find the smallest constant that can serve in (4) appears to be a rather difficult problem. The content of Theorem 1 can also be formulated in terms of an inequality not involving ε, namely as

$$d_t(J, K) \leqq \beta \sup_{H \in \mathscr{G}} d_t(J_H, K_H),$$

and the corresponding inequality for the centrally symmetric case (with β removed).

Repeated applications of Theorem 1 yields immediately the following corollary.

COROLLARY 1. *Assume that $\varepsilon \geqq 0$, $2 \leqq k < n$, and $J, K \in \mathscr{C}^n$. If for all $H \in \mathscr{H}^k$*

$$d_t(J_H, K_H) \leqq \varepsilon,$$

then

$$d_t(J, K) \leqq \beta^{n-k}\varepsilon,$$

and if both J and K are centrally symmetric, then

$$d_t(J, K) \leqq \varepsilon.$$

Obviously, a set $K \in \mathscr{C}^n$ is centrally symmetric if and only if $-K$ is a translate of K. Thus, $d_t(K, -K)$ can be used as an indicator of the asymmetry of K. Perhaps a more satisfactory indicator of this kind is the functional

$$e(K) = d_t(K, -K)/D(K), \tag{6}$$

where $D(K)$ denotes the diameter of K. This functional (which is defined only if $D(K) > 0$) has the advantage of being invariant under similarity transformations; see Grünbaum [4] for a comprehensive discussion of measures of symmetry. Theorem 1, together with the inequality $D(K_H) \leqq D(K)$, yields immediately the following stability result concerning central symmetry. K^* denotes the set obtained from K by central symmetrization, i.e., $K^* = \frac{1}{2}(K + (-K))$.

COROLLARY 2. *Let $\mathscr{G}$ be a full subset of $\mathscr{H}^{n-1}$ $(n \geqq 3)$ and $K \in \mathscr{C}^n$. If $\varepsilon \geqq 0$ and for all $H \in \mathscr{G}$*

$$d_t(K_H, -K_H) \leqq \varepsilon,$$

then

$$d_t(K, -K) \leqq \beta\varepsilon,$$

and if $e(K)$ is defined by (6) *and for all $H \in \mathscr{G}$*

$$e(K_H) \leqq \varepsilon,$$

then

$$e(K) \leqq \beta\varepsilon.$$

Furthermore, if every K_H is within Hausdorff distance ε of some centrally symmetric convex set, then K is within distance $2\beta\varepsilon$ of the centrally symmetric convex set K^*.

The last statement of this corollary can be deduced from Theorem 1 by the following argument: If for every $H \in \mathscr{G}$ we let $Z(H)$ denote the centrally symmetric set with $d(K_H, Z(H)) \leqq \varepsilon$, then $d_t(-K_H, Z(H) \leqq \varepsilon$ and these two inequalities imply $d_t(K_H^*, Z(H)) \leqq \varepsilon$. Hence,

$$d_t(K_H^*, K_H) \leqq d_t(K_H^*, Z(H)) + d_t(K_H, Z(H)) \leqq 2\varepsilon,$$

and because of Theorem 1 $d_t(K, K^*) \leqq \beta\varepsilon$. Here we have used the easily proved fact that d_t satisfies the triangle inequality, i.e.

$$d_t(A, C) \leqq d_t(A, B) + d_t(B, C) \qquad (A, B, C \in \mathscr{C}^n).$$

As an immediate consequence of Corollary 2 one obtains the known theorem that a set $K \in \mathscr{C}^n$ ($n \geqq 3$) is centrally symmetric if all its projections are (see [1, p. 125] and [6, p. 241]). Rogers [7] noted that this theorem is a consequence of the translation equivalence of two convex sets whose projections are translation equivalent. Corollary 2 shows that it is not necessary to use all projections but only those corresponding to a full subset of $\mathscr{H}^{n-1}$.

Proof of Theorem 1. According to the definition of a full subset of $\mathscr{H}^{n-1}$ we may assume that $\mathscr{G} = \tilde{\mathscr{G}} \cup \{G\}$, where $G \in \mathscr{H}^{n-1}$ and $\tilde{\mathscr{G}}$ consists of all members of $\mathscr{H}^{n-1}$ that contain a fixed $(n-2)$-space, say E, with $G^\perp \subset E$. Setting $L = G^\perp$ and performing a suitable translation we may also assume that

$$d(J_G, K_G) \leqq \varepsilon \tag{7}$$

and

$$d(J_L, K_L) \leqq \varepsilon. \tag{8}$$

There is a direction that determines supporting planes $P(J)$ and $P(K)$, of J and K (respectively) whose distance equals $d(J, K)$. If $T \in \mathscr{H}^1$ is a line orthogonal to $P(J)$ (and therefore also to $P(K)$), then there is an $H \in \mathscr{H}^{n-1}$ that contains both T and the $(n-2)$-space E. Hence, $H \in \tilde{\mathscr{G}} \subset \mathscr{G}$, and since $P(J)_H$ and $P(K)_H$ are parallel supporting planes of J_H and K_H we obtain

$$d(J, K) = d(J_H, K_H). \tag{9}$$

Since $H \in \mathscr{G}$ we have $d_t(J_H, K_H) \leqq \varepsilon$, and it follows that there is a point p in E^n such that

$$d(J_H, K_H + p) \leqq \varepsilon. \tag{10}$$

If we set $Q = G \cap H$ and let p_Q denote the image of p under the orthogonal projection of E^n onto Q, then we obtain from (7) $d(J_Q, K_Q) \leqq \varepsilon$, and from (10)

$d(J_Q, K_Q + p_Q) \leqq \varepsilon$. Using the triangle inequality for the Hausdorff distance we can infer that $d(K_Q, K_Q + p_Q) \leqq 2\varepsilon$. If we now use the fact that for every $C \in \mathscr{C}^n$ and $x \in E^n$

$$d(C, C + x) = \| x \|, \tag{11}$$

where $\| x \|$ denotes the euclidean norm of x, we obtain

$$\| p_Q \| \leqq 2\varepsilon. \tag{12}$$

Since $L \subset H$ it is clear that (10) also implies $d(J_L, K_L + p_L) \leqq \varepsilon$. Hence the triangle inequality and (8) yield $d(K_L, K_L + p_L) \leqq 2\varepsilon$ and because of (11) we can deduce that

$$\| p_L \| \leqq 2\varepsilon. \tag{13}$$

Since Q and L are orthogonal it follows from (12) and (13) that

$$\| p \| \leqq 2\sqrt{2}\varepsilon.$$

Hence, using (10), (11), and the fact that (9) implies

$$d_t(J, K) \leqq d(J_H, K_H) \leqq d(J_H, K_H + p) + d(K_H, K_H + p),$$

we find $d_t(J, K) \leqq \varepsilon + 2\sqrt{2}\varepsilon$, which is the desired result (4).

If J and K are centrally symmetric one can perform a preliminary translation so that they have the origin of E^n, say o, as a common center. As before, let H be a member of $\mathscr{G}$ such that (9) holds. Since J_H and K_H have also o as center it follows that $d(J_H, K_H + x)$, considered as a function of x, is minimal if $x = o$. Hence, $d(J_H, K_H) = d_t(J_H, K_H) \leqq \varepsilon$, and (5) follows obviously from (9).

To complete the proof of Theorem 1 we have to show that for some $n \geqq 3$ and $\varepsilon \geqq 0$ there exist two sets $J, K \in \mathscr{C}^n$ such that

$$d_t(J, K) > \varepsilon \tag{14}$$

but for all $H \in \mathscr{H}^{n-1}$

$$d_t(J_H, K_H) \leqq \varepsilon. \tag{15}$$

We take $n = 3$ and let K be the unit ball $\| x \| \leqq 1$ in E^3 and J the (regular) tetrahedron inscribed in K. If $M \in \mathscr{C}^n$ or $M \in \mathscr{C}^{n-1}$ we let $h(M, \cdot)$ denote the support function of M. Simple elementary geometric considerations show that the outer normal units vectors, say u_1, u_2, u_3, u_4, of the two-dimensional faces of J have the property that

$$(16) \qquad h(K, u_i) - h(J, u_i) = \tfrac{2}{3} \qquad (i = 1, 2, 3, 4),$$

and that for $u \neq u_i$

$$(17) \qquad 0 \leqq h(K, u) - h(J, u) < \tfrac{2}{3}.$$

Hence, $d(J, K) = \frac{2}{3}$, and since evidently $d(J + x, K) \geqq d(J, K)$ (for all $x \in E^n$) we have

$$d_t(J, K) = \tfrac{2}{3}.$$

If we now define ε by $\varepsilon = \sup\{d_t(J_H, K_H) : H \in \mathscr{H}^2\}$ we have only to show that $\varepsilon < \frac{2}{3}$. Because of obvious continuity and compactness properties it suffices to prove that for every $H \in \mathscr{H}^2$

$$(18) \qquad d_t(J_H, K_H) < \tfrac{2}{3}.$$

To prove this let H be an arbitrary but fixed plane from $\mathscr{H}^2$. Since for all $u \in H$

$$h(J, u) = h(J_H, u)$$

and

$$h(K, u) = h(K_H, u)$$

it follows from (16) and (17) that we may assume $u_1 \in H$ and therefore

$$(19) \qquad h(K_H, u_1) - h(J_H, u_1) = \tfrac{2}{3}.$$

There are two possibilities; either u_1 is the only u_i with $u_i \in H$, or there is another u_i with this property.

If $u_i \notin H$ for $i = 2, 3, 4$, then J_H is a triangle with exactly one vertex, say p, on the circle bdr K_H (namely the vertex of J_H opposite the side perpendicular to u_1). Furthermore, if v_1 and v_2 are the outer normal unit vectors belonging to the two sides meeting at p then

$$(20) \qquad h(K_H, v_i) - h(J_H, v_i) < \tfrac{2}{3} \qquad (i = 1, 2).$$

Because of (19) and (20) and the fact that the two vertices different from p are not on bdr K_H one can find a $\delta > 0$ which is so small that the set $J' = J + \delta u_1$ satisfies the inequalities

$$h(K_H, u_1) - h(J'_H, u_1) < \tfrac{2}{3} \quad \text{and} \quad h(K_H, v_i) - h(J'_H, v_i) < \tfrac{2}{3} \quad (i = 1, 2).$$

Hence, $d(J'_H, K_H) < \frac{2}{3}$ and this implies (18).

If the condition $u_i \in H$ holds not only for $i = 1$ we may assume $u_2 \in H$. In this case we have in addition to (19)

$$h(K_H, u_2) - h(J_H, u_2) = \tfrac{2}{3}, \tag{21}$$

and J_H is an isosceles triangle with exactly two vertices say p, q, on the circle bdr K_H. If v is the outer normal unit vector belonging to the side of J_H with endpoints p and q, then

$$h(K_H, v) - h(J_H, v) < \tfrac{2}{3}. \tag{22}$$

Thus, if $\delta < 0$ and if we define $J' = J + \delta(u_1 + u_2)$, then, as a consequence of (19), (21), and (22), δ can be chosen so small that $h(K_H, u_i) - h(J'_H, u_i) < \frac{2}{3}$ $(i = 1, 2)$ and $h(K_H, v) - h(J'_H, v) < \frac{2}{3}$. Hence, we find again $d(J'_H, K_H) < \frac{2}{3}$ and therefore (18).

3. Stability results concerning homothetic transformations

First we have to define suitable functions that measure the deviation of two convex sets with respect to homothetic transformations. If $J, K \in \mathscr{C}^n$ one such measure is obtained by comparing J with $\lambda K + p$ ($\lambda \geqq 0, p \in E^n$) and choosing λ and p so that the Hausdorff distance of the two sets is minimal. This leads immediately to a deviation measure, say m_1, which can be defined by

$$m_1(J, K) = \inf_{\lambda \geqq 0} d_t(J, \lambda K). \tag{23}$$

Analogously one can define

$$m_2(J, K) = \inf_{\lambda \geqq 0} d_t(\lambda J, K). \tag{24}$$

Neither $m_1(J, K)$ nor $m_2(J, K)$ is symmetric in J and K. To obtain a symmetric expression one can set

$$m(J, K) = \max\{m_1(J, K), m_2(J, K)\}. \tag{25}$$

$m_1(J, K)$ is invariant under homothetic transformations of K, and $m_2(J, K)$ under homothetic transformations of J. In some cases it is desirable to use deviation measures which are invariant under any pair of homothetic transformations that are applied, respectively, to J and K. If we assume that the diameters of J and K have the property that $D(J) > 0$ and $D(K) > 0$, then the following expressions have this invariance property:

$$(26)\qquad \tilde{m}_1(J,K) = m_1(J,K)/D(J),$$

$$(27)\qquad \tilde{m}_2(J,K) = m_2(J,K)/D(K),$$

$$(28)\qquad \tilde{m}(J,K) = \max\{\tilde{m}_1(J,K), \tilde{m}_2(J,K)\}.$$

Although it is possible to obtain stability results for arbitrary full subsets of $\mathscr{H}^{n-1}$ (see Theorem 3) we obtain more satisfactory estimates by considering special full subsets. If $K \in \mathscr{C}^n$ and $L \in \mathscr{H}^1$ we say that the line L *corresponds to the diameter of* K if $D(K_L) = D(K)$. Furthermore, a subset $\mathscr{G}$ of $\mathscr{H}^{n-1}$ will be called a *full subset of* $\mathscr{H}^{n-1}$ *associated with* K if there is a line L corresponding to the diameter of K with the following property: Either $n > 3$ and the set of those $H \in \mathscr{G}$ that contain L is already a full subset of $\mathscr{H}^{n-1}$, or $n = 3$ and $\mathscr{G}$ contains $L^\perp$ and all $H \in \mathscr{H}^2$ with $L \subset H$ (and is therefore a full subset of $\mathscr{H}^2$). The distinction between the cases $n > 3$ and $n = 3$ is necessary since for $n = 3$ there is no full subset of $\mathscr{H}^2$ with all its planes containing a fixed line.

THEOREM 2. *Let* m_1, m_2, m, $\tilde{m}_1$, $\tilde{m}_2$, $\tilde{m}$, *and* β *be defined by* (23)–(28) *and* (2). *If* $J, K \in \mathscr{C}^n$, $\varepsilon \geqq 0$, *and* $\mathscr{G}$ *is a full subset of* $\mathscr{H}^{n-1}$ *associated with* K, *then the condition*

$$(29)\qquad m_1(J_H, K_H) \leqq \varepsilon \qquad \text{for all } H \in \mathscr{G}$$

implies in the case $n > 3$

$$(30)\qquad m_1(J,K) \leqq \left(1 + \sqrt{\frac{2(n-1)}{n}}\right)\beta\varepsilon < (5 + 3\sqrt{2})\varepsilon$$

and in the case $n = 3$

$$(31)\qquad m_1(J,K) \leqq \frac{7 + 4\sqrt{3}}{3}\beta\varepsilon.$$

If K *is centrally symmetric the factors* $(1 + \sqrt{(2(n-1)/n)})\beta$ *and* $((7 + 4\sqrt{3})/3)\beta$ *can be replaced, respectively, by* 2β *and* 4β, *and if both* J *and* K *are centrally symmetric, by* 2 *and* 4.

Furthermore, if $\mathscr{G}$ *is associated with* J *(instead of* K*) the analogous relations hold for* m_2 *(instead of* m_1*), and if* $\mathscr{G}$ *is associated with both* J *and* K, *then* m_1 *can be replaced by* m. *Also, in all the above statements* m_1, m_2, m *can be replaced by* $\tilde{m}_1$, $\tilde{m}_2$, $\tilde{m}$, *respectively.*

Before we present the proof of this theorem we formulate another theorem and a corollary. The theorem concerns the situation when the given full subset of $\mathscr{H}^{n-1}$ is not necessarily associated with J or K. In this case we can still obtain stability results, but we have to assume that the convex sets do not degenerate in the sense that the ratio of the circumradius to the inradius is bounded.

THEOREM 3. *Under the same assumptions as in Theorem 2, but with $\mathscr{G}$ denoting an arbitrary full subset of $\mathscr{H}^{n-1}$, the condition* (29) *implies*

$$m_1(J,K) \leqq \left(1 + 2\frac{R(K)}{r(K)}\right)\beta\varepsilon, \tag{32}$$

where $R(K)$ denotes the circumradius and $r(K)$ the inradius of K. If both J and K are centrally symmetric the factor β can be omitted.

Furthermore, all analogous relations with m_1 replaced by m_2 or m (and correspondingly $R(K)/r(K)$ by $R(J)/r(J)$ or $\max\{R(J)/r(J), R(K)/r(K)\}$*) and the corresponding statements for $\bar{m}_1$. $\bar{m}_2$, and $\bar{m}$ are true.*

As a noteworthy special case we consider a stability version of the well-known theorem that a convex body is a ball if all its projections are. Let us assume that $K \in \mathscr{C}^n$ is arbitrary and J is a ball in E^n. If $\mathscr{G}$ is a full subset of $\mathscr{H}^{n-1}$ and if, for all $H \in \mathscr{G}$, $m_2(J_H, K_H) \leqq \varepsilon$, then one obtains from Theorem 2 (if $n > 3$) and Theorem 3 (if $n = 3$) $m_2(J,K) \leqq 2\beta\varepsilon$ and $m_2(J,K) \leqq 3\beta\varepsilon$, respectively. This implies obviously $R(K) - r(K) \leqq 4\beta\varepsilon$ ($n > 3$) and $R(K) - r(K) \leqq 6\beta\varepsilon$ ($n = 3$). If we also note that from $R(K_H) - r(K_H) \leqq \varepsilon$ there follows $m_2(J_H, K_H) \leqq \varepsilon$ we obtain immediately the following

COROLLARY 3. *Let $\mathscr{G}$ be a full subset of $\mathscr{H}^{n-1}$ ($n \geqq 3$), and let the set $K \in \mathscr{C}^n$ have circumradius $R(K)$ and inradius $r(K)$. If an $\varepsilon \geqq 0$ is given and if for every $H \in \mathscr{G}$ the projection K_H is within (Hausdorff) distance ε of an $(n-1)$-dimensional ball then K is within distance $2\beta\varepsilon$ (if $n > 3$) and $3\beta\varepsilon$ (if $n = 3$) of a ball in E^n. Furthermore, if a $K \in \mathscr{C}^n$ has the property that for all $H \in \mathscr{G}$*

$$R(K_H) - r(K_H) \leqq \varepsilon,$$

then

$$R(K) - r(K) \leqq 4\beta\varepsilon \qquad (n > 3)$$

and

$$R(K) - r(K) \leqq 6\beta\varepsilon \qquad (n = 3).$$

In all these statements the factor $\beta = 1 + 2\sqrt{2}$ can be removed if K is centrally symmetric.

PROOF OF THEOREM 2. Because of $m_2(J, K) = m_1(K, J)$ it is obvious that the statements regarding m_2 and m follow from those concerning m_1. Also, since for any $H \in \mathscr{H}^{n-1}$ and $C \in \mathscr{C}^n$ we have $D(C_H) \leqq D(C)$ the statements regarding $\bar{m}_1$, $\bar{m}_2$, $\bar{m}$ follow immediately from the corresponding assertions about m_1, m_2, m. Thus, it suffices to prove only the statements involving m_1.

Let $L \in \mathscr{H}^1$ be a line corresponding to a diameter of K, and let $H \in \mathscr{G}$ be such that $L \subset H$. It is convenient to set

$$D_0 = D(K).$$

After applying a preliminary homothetic transformation to K we may assume that

$$D(J_L) = D_0. \tag{33}$$

Because of (29) and $H \in \mathscr{G}$ there is a $\lambda \geqq 0$ such that

$$d_t(J_H, \lambda K_H) \leqq \varepsilon. \tag{34}$$

Let now K_H^o denote the translate of K_H whose circumsphere center is o, and let $h(K_H^o, u)$ be the support function of K_H^o. Then we have

$$\begin{aligned} d_t(\lambda K_H, K_H) \leqq d(\lambda K_H^o, K_H^o) &\leqq \sup_u |h(\lambda K_H^o, u) - h(K_H^o, u)| \\ &\leqq |\lambda - 1| \sup_u h(K_H^o, u). \end{aligned}$$

Using Jung's inequality (see [1, p. 78]) we find

$$\sup_u h(K_H^o, u) = R(K_H^o) \leqq \sqrt{\frac{n-1}{2n}}\, D(K_H^o),$$

and therefore

$$d_t(\lambda K_H, K_H) \leqq |\lambda - 1| \sqrt{\frac{n-1}{2n}}\, D(K_H^o) \leqq |\lambda - 1| \sqrt{\frac{n-1}{2n}}\, D_0. \tag{35}$$

From (34), (35), and the triangle inequality one obtains

$$d_t(J_H, K_H) \leqq d_t(J_H, \lambda K_H) + d_t(\lambda K_H, K_H) \leqq \varepsilon + \sqrt{\frac{n-1}{2n}}\, |\lambda - 1| D_0.$$

Since (34) and $L \subset H$ imply

$$d_t(J_L, \lambda K_L) \leqq \varepsilon,$$

we obtain from (33) and the fact that $D(\lambda K_L) = \lambda D_0$

$$|\lambda - 1| D_0 \leqq 2\varepsilon.$$

Thus,

$$d_t(J_H, K_H) \leqq \left(1 + \sqrt{\frac{2(n-1)}{n}}\right)\varepsilon. \tag{36}$$

If $n > 3$ the line L can be chosen so that the collection of all $H \in \mathscr{G}$ with $L \subset H$ is a full subset of $\mathscr{H}^{n-1}$. Theorem 1 yields therefore

$$m_1(J, K) \leqq d_t(J, K) \leqq \left(1 + \sqrt{\frac{2(n-1)}{n}}\right)\beta\varepsilon,$$

which is the desired inequality (30). If K is centrally symmetric, then $\sup_u h(K_H^o, u) = \frac{1}{2}D(K_H^o)$ and (35) can be replaced by

$$d_t(\lambda K_H, K_H) \leqq |\lambda - 1| \tfrac{1}{2} D_0.$$

Following the same procedure as before this enables one to deduce $d_t(J_H, K_H) \leqq 2\varepsilon$. Using Theorem 1 we find therefore $m_1(J, K) \leqq 2\beta\varepsilon$ or, if J is also centrally symmetric, $m_1(J, K) \leqq 2\varepsilon$.

In the case $n = 3$ the situation is slightly more complicated. We may assume that $\mathscr{G} = \{L^\perp\} \cup \tilde{\mathscr{G}}$, where $\tilde{\mathscr{G}} = \{H : H \in \mathscr{H}^2, L \subset H\}$ and $D(J_L) = D(K) = D_0$. Then, (36) shows that for any $H \in \tilde{\mathscr{G}}$

$$d_t(J_H, K_H) \leqq (1 + 2/\sqrt{3})\varepsilon$$

and this implies obviously that for any line $Q \subset H$

$$d_t(J_Q, K_Q) \leqq (1 + 2/\sqrt{3})\varepsilon. \tag{37}$$

We note now that because of (29) there is a $\mu \geqq 0$ such that

$$d_t(J_{L^\perp}, \mu K_{L^\perp}) \leqq \varepsilon. \tag{38}$$

Taking $Q = L^\perp \cap H$ we have therefore

$$d_t(J_Q, \mu K_Q) \leqq \varepsilon. \tag{39}$$

Using (37), (39), and the triangle inequality we find

$$d_t(\mu K_Q, K_Q) \leqq 2(1 + 1/\sqrt{3})\varepsilon. \tag{40}$$

Let $K^o_{L^\perp}$ be the translate of $K_{L^\perp}$ that has o as circumsphere center. Then, using the same argument that lead to (35) we find

$$d(\mu K^o_{L^\perp}, K^o_{L^\perp}) \leqq |\mu - 1| \frac{1}{\sqrt{3}} D(K^o_{L^\perp}). \tag{41}$$

Since (40) implies

$$|\mu - 1| D(K_Q) \leqq 4(1 + 1/\sqrt{3})\varepsilon,$$

and since Q can be any line in $L^\perp$ this shows that

$$|\mu - 1| D(K^o_{L^\perp}) \leqq 4(1 + 1/\sqrt{3})\varepsilon.$$

From this inequality and (41) we obtain

$$d(\mu K^o_{L^\perp}, K^o_{L^\perp}) \leqq \tfrac{4}{3}(1 + \sqrt{3})\varepsilon. \tag{42}$$

Consequently, if (38) is combined with (42) we find

$$d_t(J_{L^\perp}, K_{L^\perp}) \leqq d_t(J_{L^\perp}, \mu K_{L^\perp}) + d_t(\mu K_{L^\perp}, K_{L^\perp}) \leqq \tfrac{1}{3}(7 + 4\sqrt{3})\varepsilon.$$

Using this and (36) we can state that for all $G \in \mathscr{G}$

$$d_t(J_G, K_G) \leqq \tfrac{1}{3}(7 + 4\sqrt{3})\varepsilon.$$

Hence Theorem 1 yields immediately the desired inequality (31).

If K is centrally symmetric, then, as already remarked, (36) can be replaced by $d_t(J_H, K_H) \leqq 2\varepsilon$. This implies that (40) holds with the constant 3 instead of $2(1 + 1/\sqrt{3})$. Since it is also clear that in this case the factor $1/\sqrt{3}$ in (41) can be replaced by $\frac{1}{2}$ one obtains the coefficient 3 (instead of $\frac{4}{3}(1 + \sqrt{3})$) in (42), and this leads immediately to $d_t(J_G, K_G) \leqq 4\varepsilon$ for all $G \in \mathscr{G}$. Thus, Theorem 1 shows that $m_1(J, K) \leqq 4\beta\varepsilon$, and that the factor β can be omitted if J is also centrally symmetric.

Proof of Theorem 3. Again it suffices to consider only the deviation measure m_1. As in the proof of Theorem 1 we may suppose that $G = \tilde{\mathscr{G}} \cup \{G\}$ where $G \in \mathscr{H}^{n-1}$, and $H \in \tilde{\mathscr{G}}$ if and only if $E \subset H$ where E is a fixed $(n-2)$-space with $G^\perp \subset E$. Also, in view of (29) we may assume that K has already been subjected to a preliminary homothetic transformation so that

$$d(J_G, K_G) \leqq \varepsilon. \tag{43}$$

As another consequence of (29) we note that for every $H \in \tilde{\mathscr{G}}$ there is a $\lambda \geqq 0$ such that

(44) $$d_t(J_H, \lambda K_H) \leqq \varepsilon.$$

Thus, if U is a line in $G \cap H$ we have $d(J_U, K_U) \leqq \varepsilon$ and $d_t(J_U, \lambda K_U) \leqq \varepsilon$. An obvious application of the triangle inequality yields therefore

$$d_t(\lambda K_U, K_U) \leqq 2\varepsilon.$$

Since this implies $|D(\lambda K_U) - D(K_U)| \leqq 4\varepsilon$, and since $D(K_U) \geqq 2r(K)$ we obtain

(45) $$|\lambda - 1| \leqq 2\varepsilon / r(K).$$

Again, let K_H^o denote the translate of K_H with circumsphere center at o. Then,

$$d_t(\lambda K_H, K_H) \leqq d(\lambda K_H^o, K_H^o) \leqq \sup_u |h(\lambda K_H^o, u) - h(K_H^o, u)|$$
$$\leqq |\lambda - 1| \sup_u h(K_H^o, u) \leqq |\lambda - 1| R(K_H).$$

Because of (45) this shows that

$$d_t(\lambda K_H, K_H) \leqq 2 \frac{R(K)}{r(K)} \varepsilon.$$

Combining this with (44) we find

$$d_t(J_H, K_H) \leqq d_t(J_H, \lambda K_H) + d_t(\lambda K_H, K_H) \leqq \varepsilon + 2 \frac{R(K)}{r(K)} \varepsilon.$$

Theorem 3 is now an immediate consequence of this inequality, (43), and Theorem 1.

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